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The operator method of the approximate description of the quantum and classical systems

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Abstract. Systematic theory of the operator method (OM) of the approximate solution of the Schrödinger equation is considered. The eigenfunctions and eigenvalues for various one- and many-dimensional systems are calculated and the arguments of the convergence of the OM perturbation series are discussed. It is also shown that OM permits us to develop the absolute convergent approximation in the theory of nonlinear oscillations of the classical systems.

1. Introduction

At present the interest in the construction of non-perturbation methods of description of the quantum systems is increasing. One of such methods called the operator method (OM) was recently introduced by Feranchuk and Komarov (1982a) on the quartic anharmonic oscillator (QAO) example. Fernandez and Castro (1982) found the simple model system where one could essentially increase the radius of convergence of the OM approximation series in comparison with the perturbation theory series. Gerry and Silverman (1983) connected the OM with the group characteristics of the annihilation and creation operators. Feranchuk and Komarov (1982b) and Witschel (1983) used the OM for systems with infinite number of degrees of freedom and recently Yamazaki (1984) described in detail the same results for QAO which we found earlier (Feranchuk and Komarov 1982a).

The main merit of the OM in the QAO problem is that even in zeroth approximation it gives very simple but sufficiently precise estimation for eigenvalues and eigenfunctions of the ground and excited states in the whole range of the anharmonicity constant λ. The high-order approximations of the OM are calculated by means of the regular procedure and they give the series which doesn't contain any small algebraic parameter and is rapidly convergent for all λ. Such a situation is very unusual in physical problems and therefore the analysis of the reasons for OM convergence is of great interest. Also, it is very important to ascertain what is the generality of the results obtained in the partial problem of the QAO. These questions are considered in the present work.

Our paper has the following structure. Modifications of the OM which enable us to improve its convergence and to find a singular point of an energy spectrum are considered in § 2. It is also shown in this section that the OM permits one to calculate the continuous spectrum wavefunctions. The questions connected with the OM convergence are discussed in § 3 using the example of the QAO problem. Specifically, it is shown that the series obtained in this problem on the basis of the OM is absolutely

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convergent while the perturbation theory series is asymptotic and its radius of convergence is equal to zero. The OM series proves to be rapidly convergent if the special parameter is chosen in an optimal way. The systems with additional integral of motion are considered in § 4. As we will see below, the OM permits one to calculate the energy levels of such systems by means of sufficiently simple formulae but with high accuracy.

Next we develop a new method in the theory of nonlinear oscillations of classical systems on the basis of the OM (§ 5). This method is a non-asymptotical one and it defines the solution of the nonlinear equation of motion for arbitrary oscillation amplitude.

Lastly, utilisation of the OM for many-dimensional systems is considered in § 6 and the energy spectra of coupled quartic anharmonic oscillators and the hydrogen atom in the uniform magnetic field are calculated.

2. Modification of the operator method

We shall not consider in detail the original formulation of the OM described in the paper of Feranchuk and Komarov (1982a) which we shall refer to as FK. According to this paper, the Hamiltonian $\hat{\mathscr{H}}(x, \hat{p}, \lambda)$ of the arbitrary system must be put in second quantised form through the introduction of creation a^+ and annihilation a operators

$$x = (a + a^+ + 2u)/(2\omega)^{1/2}, \qquad \hat{p} = \mathrm{i}(\omega/2)^{1/2}(a^+ - a), \qquad [a, a^+] = 1 \tag{1}$$

with arbitrary parameters ω and u.

Then $\hat{\mathscr{H}}(a, a^+, \lambda)$ is divided into two parts

$$\hat{\mathscr{H}}(a, a^+, \lambda) = \hat{\mathscr{H}}_0(\hat{n}, \omega, u, \lambda) + \hat{V}(a^+, a, \omega, u, \lambda), \tag{2}$$

where $\hat{\mathscr{H}}_0$ contains all terms which commute with the particle number operator $\hat{n} = a^+ a$, consequently its eigenvalues and eigenfunctions are easily calculated. It was shown that perturbation theory with respect to the operator $\hat{V}$ in the QAO problem leads to rapidly convergent series for all eigenvalues $E_n(\lambda)$ and coupling constant λ. The parameters ω_n and u_n were chosen by FK in such a way that $E_n^{(0)}(\omega_n u_n)$ was a minimum, where $E_n^{(0)}(\omega_n u_n)$ was given by the equations

$$\hat{\mathscr{H}}_0|n\rangle = E_n^{(0)}(\omega_n u_n)|n\rangle, \qquad a^+ a|n\rangle = n|n\rangle,$$
$$\partial E_n^{(0)}/\partial \omega_n = \partial E_n^{(0)}/\partial u_n = 0. \tag{3}$$

However, this form of the OM is suitable for discrete spectrum states only and is inapplicable for a Hamiltonian where the bound states disappear for the critical value λ_c or for a description of the continuous spectrum states. Therefore let us introduce a modification of the OM which permits us to consider the pointed cases too.

If the Hamiltonian $\hat{\mathscr{H}}$ has no bound states for $\lambda < \lambda_c$, then it proves that equation (3) for the parameters $u_n^{(0)}, \omega_n^{(0)}$ has any real-valued solution for $\lambda < \lambda_c^{(0)}$ (see the problem described later). The value $\lambda_c^{(0)}$ gives a zeroth-order approximation for critical value λ_c. However, using perturbation theory with respect to $\hat{V}(\omega_n^{(0)}, u_n^{(0)})$ in the form described by FK doesn't permit us to find λ_c exactly because equation (3) and parameters $u_n^{(0)}, \omega_n^{(0)}$ are invariable in higher orders. A new scheme for calculation of the parameters u_n, ω_n is based on the following arguments. If the zeroth-approximation state vector $|n\rangle$ satisfies the condition $\hat{V}|n\rangle = 0$, it would be an exact eigenfunction of $\hat{\mathscr{H}}$. As in the

definition the operator $\hat{V}$ changes the particle number n, it may be represented as follows

$$\hat{V} = a^{+}\hat{v}_1^{(0)}(\omega, u, \hat{n}) + a^{+2}\hat{v}_2^{(0)}(\omega, u, \hat{n}) + \text{HC} + \ldots. \tag{4}$$

Now let us choose the parameters $\omega_n^{(0)}$, $u_n^{(0)}$ such a way that the condition $\hat{V}|n\rangle = 0$ holds as accurately as possible. This leads to the following zeroth-order equations

$$\hat{v}_1^{(0)}(\omega, u, n) = \hat{v}_2^{(0)}(\omega, u, n) = 0, \tag{5}$$

which define $\omega_n^{(0)}$ and $u_n^{(0)}$. It may be easily verified that equations (3) and (5) are equivalent to each other for the ground state ($n = 0$).

In order to find more precise values $\omega_n^{(1)}$ and $u_n^{(1)}$ one can use the first-order correction to the Hamiltonian eigenfunction. Then the condition $\hat{V}|n\rangle = 0$ transforms as

$$\hat{V}|\psi_n^{(1)}\rangle = \hat{V}_1(\omega, u)|n\rangle \equiv [\hat{V} + \hat{V}(\hat{\mathscr{H}}_0 - E_n^{(0)})^{-1}\hat{V}]|n\rangle. \tag{6}$$

The operator $\hat{V}_1$ has a form analogous to (4):

$$\hat{V}_1(\omega, u) = a^{+}\hat{v}_1^{(1)}(\omega, u, \hat{n}) + a^{+2}\hat{v}_2^{(1)}(\omega, u, \hat{n}) + \text{HC} + \ldots$$

and one can find the first-order equations for the parameters $\omega_n^{(1)}$ and $u_n^{(1)}$

$$v_1^{(1)}(\omega, u, n) = v_2^{(1)}(\omega, u, n) = 0. \tag{7}$$

It is evident that analogous equations may be built in any order with respect to $\hat{V}$.

Let us use this form of the OM in two problems. In the case of QAO the operators $\hat{\mathscr{H}}_0$ and $\hat{V}$ were written in FK

$$\hat{\mathscr{H}}_0 = \tfrac{1}{4}(\omega + 1/\omega)(2\hat{n} + 1) + (3\lambda/4\omega^2)(1 + 2\hat{n} + 2\hat{n}^2),$$
$$\hat{V} = \tfrac{1}{4}(1/\omega - \omega)(a^{+2} + a^2) + (\lambda/4\omega^2)[6(a^2 + a^{+2}) + a^4 + a^{+4} + 4(a^{+}a^3 + a^{+3}a)]. \tag{8}$$

Zeroth-order operators $\hat{v}_1^{(0)}$ and $\hat{v}_2^{(0)}$ are equal to

$$\hat{v}_1^{(0)} \equiv 0, \qquad u = 0, \qquad \hat{v}_2^{(0)} = \tfrac{1}{4}(1/\omega - \omega) + \lambda/\omega^2(\tfrac{3}{2} + \hat{n}).$$

and the equation for $\omega_n^{(0)}$ is

$$\omega^3 - \omega - 2\lambda(2n + 3) = 0. \tag{9}$$

The variational principle (3) used by FK leads to the different equation which coincides with (9) only for $n = 0, 1$.

Table 1 compares the second excited state energy $E_{2,M}^{(0)}$, calculated in zeroth approximation with $\omega_n^{(0)}$ from equation (9), and second-order approximation $E_2^{(0)} + E_2^{(2)}$ found by FK by means of (3). These results show that the modification considered improves the precision of the OM calculations.

Table 1. Energy levels of the anharmonic oscillator.

	λ		
E	0.1	1.0	10
E_2 (accurate)	3.138 62	5.179 29	10.3471
$E_2^{(0)} + E_2^{(2)}$ (FK)	3.139 63	5.187 51	10.3716
$E_{2M}^{(0)}$ (9)	3.139 34	5.177 32	10.3371
E_0 (accurate)	0.559 146	0.803 771	1.504 97
$E_0^{(0)} + E_0^{(2)}$ (FK)	0.561 172	0.805 734	1.509 35
$E_0^{(0)} + E_{2M}^{(0)}$ (10)	0.559 199	0.804 351	1.507 21

It may be noticed that the ground-state energy E_0 is also calculated more precisely in comparison with FK, if one takes into account the first-order equations (7). In the case considered these equations are

$$\omega^3 - \omega - 6\lambda - \tfrac{3}{2}\lambda(\omega^3 - \omega - 14\lambda)/(\omega^3 + \omega + 15\lambda) = 0 \tag{10}$$

and the values $E^{(0)}_{0,M} + E^{(2)}_{0,M}$ and $E^{(0)}_0 + E^{(2)}_0$ are also listed in table 1.

Let us now consider the calculation of the ground-state energy for a particle in the screened Coulomb potential:

$$\mathcal{H} = \tfrac{1}{2}p^2 - e^{-\lambda r}/r. \tag{11}$$

The bound eigenstates of such a Hamiltonian disappear when $\lambda > \lambda_c$ and we shall find λ_c by means of the OM. Komarov and Romanova (1982) showed that the most convenient presentation of the Hamiltonian with Coulomb singularity of the potential is given by the following coordinate transformation

$$x_\lambda = \xi^*_s(\sigma_\lambda)_{st}\xi_t, \qquad r = \xi^*_s\xi_s, \qquad s, t = 1, 2,$$

where σ_λ are the Pauli matrices; ξ_s are the complex-valued variables which form a spinor. Using the second quantised form

$$a_s = (\omega/2)^{1/2}\left(\xi_s + \frac{1}{\omega}\frac{\partial}{\partial \xi^*_s}\right), \qquad a^+_s = (\omega/2)^{1/2}\left(\xi^*_s - \frac{1}{\omega}\frac{\partial}{\partial \xi_s}\right), \qquad [a_s, a^+_t] = \delta_{st};$$

$$b_s = (\omega/2)^{1/2}\left(\xi^*_s + \frac{1}{\omega}\frac{\partial}{\partial \xi_s}\right), \qquad b^+_s = (\omega/2)^{1/2}\left(\xi_s - \frac{1}{\omega}\frac{\partial}{\partial \xi^*_s}\right), \qquad [b_s, b^+_t] = \delta_{st},$$

one can build the following operator, put in the normal form

$$\begin{aligned}\hat{\mathcal{L}} = r(\hat{H} - E) &= \tfrac{1}{4}\omega(2 - M^+ + N - M) - \tfrac{1}{2}(E/2\omega)(2 + M^+ + N + M)\\ &\quad -(1+\mu)^{-2}\exp\{-[\mu/(1+\mu)]M^+\}\exp\{-N\ln(1+\mu)\}\\ &\quad \times\exp\{-[\mu/(1+\mu)]M\},\\ M &= a_s b_s, \qquad N = a^+_s a_s + b^+_s b_s, \qquad \mu = \lambda/2\omega.\end{aligned}$$

Then the eigenfunctions of the Hamiltonian (11) are such solutions of the equation

$$\mathcal{L}|\psi_\varepsilon\rangle = \varepsilon|\psi\rangle \tag{12}$$

which correspond to $\varepsilon = 0$.

Let us use the OM to solve equation (12). The zeroth-order operator $\hat{\mathcal{L}}_0$ is

$$\hat{\mathcal{L}}_0 = \left(\frac{\omega}{4} - \frac{E}{2\omega}\right)(2 + \hat{N}) - (1+\mu)^{-2}\sum_{s=0}^{\infty}\frac{1}{(s!)^2}\left(\frac{\mu}{1+\mu}\right)^{2s}(M^+)^s\exp[-N\ln(1+\mu)]M^s,$$

and the perturbation operator is

$$\begin{aligned}\hat{\mathcal{L}}_1 = &-\left(\frac{\omega}{4} + \frac{E}{2\omega}\right)(M + M^+)\\ &-(1+\mu)^{-2}\exp\left(-\frac{\mu}{1+\mu}M^+\right)\exp[-N\ln(1+\mu)]\exp\left(-\frac{\mu}{1+\mu}M\right)\\ &+(1+\mu)^{-2}\sum_{s=0}^{\infty}\frac{1}{(s!)^2}\left(\frac{\mu}{1+\mu}\right)^{2s}(M^+)^s\exp[-N\ln(1+\mu)]M^s.\end{aligned}$$

In zeroth approximation

$$|\psi_\varepsilon^{(0)}\rangle = |0\rangle, \qquad a_s|0\rangle = b_s|0\rangle = 0,$$

and values $E_0^{(0)}$, $\omega_0^{(0)}$ are defined by the following equations

$$\varepsilon_0^{(0)} = 2A - 1/(1+\mu)^2 = 0, \qquad A = \tfrac{1}{4}\omega - \tfrac{1}{2}E/\omega,$$

$$[-B + \mu/(1+\mu)^3]M^+|0\rangle = 0, \qquad B = \tfrac{1}{4}\omega + \tfrac{1}{2}E/\omega.$$

One can find $\lambda_c^{(0)}$ from the condition $E_0^{(0)} = 0$, which leads to a rough estimation of $\lambda_c^{(0)} = 1$, essentially different from the value calculated, for example, by Sergeev and Sherstiuk (1982).

Let us now take into account the second-order correction to ε_0 and the first-order correction to the state vector $|\psi_\varepsilon\rangle$ in order to build the concrete form of general equations (7) in the case considered. After simple but unwieldy algebraic transformation the equations may be written as follows:

$$2A - (1+\mu)^{-2} = 2[B - \mu(1+\mu)^{-3}]^2[2A + (1+\mu)^{-2} - (1+2\mu^2)(1+\mu)^{-4}]^{-1} + \frac{1}{(1+\mu)^4}\sum_{n=2}^{\infty}\left(\frac{\mu}{1+\mu}\right)^{2n}\frac{n+1}{2An+z_n},$$

$$B\{1+3\mu^2(1+\mu)^{-4}[4A + (1+\mu)^{-2} - (1+6\mu^2+3\mu^4)(1+\mu)^{-6}]^{-1}\} = \mu(1+\mu)^{-3} + \frac{1}{2\mu(1+\mu)^5}\sum_{n=2}^{\infty}\left(\frac{\mu}{1+\mu}\right)^{2n}\frac{(n+1)(n+2\mu^2)}{2An+z_n}, \tag{13}$$

$$z_n = (1+\mu)^{-2} - (1+\mu)^{-2(n+1)}\sum_{s=0}^{n}\mu^{2s}\frac{n!(n+1)!}{(s!)^2(n-s)!(n+1-s)!}.$$

It was discussed above, that in the considered modification of the OM, the parameter $\omega_0^{(1)}$ from equation (13) is not equal to $\omega_0^{(0)}$. Table 2 lists the calculation results for function $E_0(\lambda)$, found in zeroth and second approximations. One can see that the considered scheme of the OM gives rapidly convergent series for λ_c.

Use of the OM for an approximate description of continuous spectrum wavefunctions is also of great interest. It proves that one can solve this problem using the analytical continuation of the parameters ω and n to the complex-valued plane. In order to explain the main idea of such a calculation, let us consider the following simple Hamiltonian:

$$\hat{\mathscr{H}} = \tfrac{1}{2}(\hat{p}^2 - \hat{x}^2). \tag{14}$$

This Hamiltonian possesses the continuous spectrum only.

Direct use of the OM leads to the equation

$$[\tfrac{1}{4}\omega(2a^+a + 1 - (a^+)^2 - a^2) - (1/4\omega)(2a^+a + 1 + (a^+)^2 + a^2) - \varepsilon]|\psi\rangle = 0,$$

Table 2. Ground-state energy in the screened Coulomb potential ($\lambda_c = 1.19$ (Sergeev 1982)).

			λ		
E_0	0	0.6	1.0	1.15	1.195
$\lvert E_0^{(0)}\rvert$	0.5	0.103	0	—	—
$\lvert E_0^{(0)} + E_0^{(2)}\rvert$	0.5	0.116	0.013	0.0017	0

and one can obtain the following results

$$|\psi_n^{(0)}\rangle = |n, \omega_n\rangle, \qquad \omega_n = \mathrm{i}, \qquad \varepsilon_n = \mathrm{i}(n+\tfrac{1}{2}). \tag{15}$$

The values ε_n from (15) will coincide with the real-valued energy E for the state vector $|\psi_n^{(0)}\rangle$, corresponding to the complex-valued number $n = -\mathrm{i}E - \frac{1}{2}$. In order to continue $|\psi_n^{(0)}\rangle$ to the complex-valued n, one can use the following equation

$$a_\omega^+ |n, \omega\rangle = (n+1)^{1/2} |n+1, \omega\rangle, \tag{16}$$

which defines the matrix element of the creation operator when n is the positive integer. But (16) may be also solved for complex-valued n in the form of the following contour integral

$$|\psi(n, \omega)\rangle = \frac{1}{2\pi\mathrm{i}}[\Gamma(n+1)]^{1/2} \int_C \mathrm{d}t \frac{\mathrm{e}^{ta_\omega^+}}{t^{n+1}} |0, \omega\rangle, \tag{17}$$

where $\Gamma(z)$ is the gamma function; the vacuum state is defined as follows

$$a_\omega |0, \omega\rangle = 0$$

and the contour C in the complex-valued plane t is shown in figure 1.

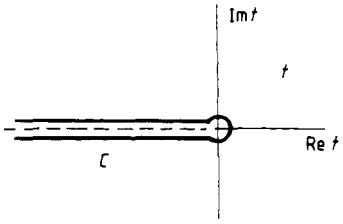

Figure 1. Countour of the integration in equation (20).

According to (15), the vacuum state $|0, \omega\rangle$ and the operators a_ω and a_ω^+ correspond to pure imaginary frequency ω. In order to define these values let us start from the Hamiltonian of the usual oscillator with frequency $\omega = 1$

$$\hat{\mathscr{H}} = \tfrac{1}{2}(\hat{p}^2 + \hat{x}^2).$$

The second quantised form and the ground state of $\hat{\mathscr{H}}$ are well known

$$a = (1/\sqrt{2})(\hat{x} + \mathrm{i}\hat{p}), \qquad a^+ = (1/\sqrt{2})(\hat{x} - \mathrm{i}\hat{p}), \qquad \hat{\mathscr{H}} = a^+ a + \tfrac{1}{2}, \qquad a|0\rangle = 0. \tag{18}$$

It proves that one can transform the values in (18) to arbitrary frequency ω by means of the following operator equations

$$a_\omega = \hat{R}(\omega, 1) a \hat{R}^{-1}(\omega, 1), \qquad |0, \omega\rangle = \hat{R}(\omega, 1)|0\rangle,$$

$$\hat{R}(\omega, \omega') = \exp\left(\frac{1}{4} \ln \frac{\omega}{\omega'} (A_{\omega'}^+ - A\omega')\right), \tag{19}$$

$$A_{\omega'} = a_{\omega'}^2,$$

where the normal form of the operator $\hat{R}$ is

$$\hat{R}(\omega, \omega') = \exp(\varphi_1 A_{\omega'}^+) \exp[\varphi_2(N_{\omega'} + \tfrac{1}{2})] \exp(-\varphi_1 A_{\omega'})$$

with

$$\varphi_1 = \frac{\omega' - \omega}{2(\omega + \omega')}, \qquad \varphi_2 = \ln \frac{2(\omega\omega')^{1/2}}{(\omega + \omega')}, \qquad N_{\omega'} = a^+_{\omega'} a_{\omega'}.$$

Equation (19) permits us to build the sought analytical continuation to the complex-valued ω.

Let us substitute equation (19) with $\omega = \mathrm{i}$ in (17), put $n = -\mathrm{i}E - \frac{1}{2}$ and transform all operators in the normal form. Then one can find the following expression for the continuous spectrum state vector, corresponding to energy E

$$|\psi_E\rangle = \frac{1}{2\pi\mathrm{i}}[\Gamma(-\mathrm{i}E + \tfrac{1}{2})]^{1/2} \int_C \frac{\mathrm{d}t}{\sqrt{t}} \exp[\mathrm{i}(E \ln t + t^2)] \exp(\sqrt{2}ta^+)|0\rangle. \qquad (20)$$

It is easy to ensure that equation (20) gives the operator integral representation of the confluent hypergeometric function which is the exact eigenfunction of the Hamiltonian (14).

3. Convergence of the OM

The results described in the previous section and in the following ones demonstrate a rapid convergence of the series obtained on the basis of the OM. This convergence is observed for all eigenstates and in the whole range of the Hamiltonian parameters including the most complicated intermediate coupling regime. The algorithm of the calculation of the high-order corrections is sufficiently simple although this expansion is not connected with any algebraic parameter. Even zeroth approximation defines eigenvalues, as a rule, with a relative accuracy to order of one percent and sometimes it gives a somewhat unexpected accuracy (see, for example, §§ 4–5).

Strict analysis of the OM convergence is apparently a complicated mathematical problem and its solution falls outside the framework of this article. Therefore we shall consider several simple arguments which don't give a strict proof but do explain to some extent the convergence of the OM.

Firstly let us discuss shortly a simple exhibit built by Fernandez and Castro (1982). They considered the following Hamiltonian

$$\hat{\mathscr{H}} = \tfrac{1}{2}(\hat{p}^2 + \hat{x}^2) + \lambda\hat{x}^2. \qquad (21)$$

Its eigenvalues are obvious and exist for any $\lambda > -\frac{1}{2}$

$$E_n = (n + \tfrac{1}{2})(1 + 2\lambda)^{1/2}.$$

However, if one calculates E_n by means of canonical perturbation theory (CPT) as regards the operator $\lambda\hat{x}^2$, the series in terms of powers of λ with radius of convergence $|\lambda| < \frac{1}{2}$ will be obtained. It is conditioned by the branch point of the function $E_n(\lambda)$ in the complex-valued plane.

Now let us put the Hamiltonian (21) into the second quantised form by means of the (1) with $u = 0$

$$\hat{\mathscr{H}} = \frac{1}{4}\left(\frac{1 + 2\lambda}{\omega} - \omega\right)((a^+)^2 + a^2) + \frac{1}{2}\left(\frac{1 + 2\lambda}{\omega} + \omega\right)(a^+a + \tfrac{1}{2}). \qquad (22)$$

Use of the OM consists of two essential points: (a) the introduction all terms commuted

with the operator $\hat{n} = a^+ a$ to the zeroth-order Hamiltonian, and (b) the choice of the parameter ω in order to obtain the best zeroth approximation. One can investigate the role of each of these points using the Hamiltonian (22). Firstly let us choose the frequency ω arbitrarily and put it equal to 1. Then the zeroth-order Hamiltonian is

$$\hat{\mathcal{H}}_0 = (1+\lambda)(\hat{n} + \tfrac{1}{2})$$

and the perturbation operator is

$$\hat{\mathcal{H}}_1 = \tfrac{1}{2}\lambda((a^+)^2 + a^2).$$

Then it is obviously that the perturbation theory in respect of $\hat{\mathcal{H}}_1$ gives the oscillating series in terms of powers of the parameter

$$\xi = \lambda^2/(1+\lambda)^2$$

and this series is convergent for all $-\frac{1}{2} < \lambda < \infty$.

Thus, point (a) leads to the definite value of expansion parameter ξ for any amplitude of the perturbation operator. It is conditioned by the appearance of the parameter λ in the denominator of terms of the OM series due to the propagator $(\hat{\mathcal{H}}_0 - E_n^{(0)})^{-1}$. This result is general and essential for the OM convergence in any problem. Use of point (b) of the OM will lead to the accurate eigenvalues of the considered Hamiltonian (22) if one chooses the parameter ω in accordance with equations (5). This result has a special character, of course, but we shall see later that the particular choice of ω in the general case improves essentially the convergence of the OM series too because many of its terms become zero.

Let us now consider the convergence of the OM series for the problem of the ground-state energy of the QAO. According to FK the zeroth-order Hamiltonian with arbitrary parameter ω is

$$\hat{\mathcal{H}}_0 = \frac{1}{4}\left(\omega + \frac{1}{\omega}\right)(2\hat{n}+1) + \frac{3\lambda}{4\omega^2}(4\hat{n} + 2(a^+)^2 a^2 + 1),$$

and the perturbation operator is

$$\hat{\mathcal{H}}_1 = \tfrac{1}{4}(1/\omega - \omega)((a^+)^2 + a^2) + (\lambda/4\omega^2)((a^+)^4 + a^4 + 4(a^+)^3 a + 4a^+ a^3 + 6(a^+)^2 + 6a^2)$$

and differs from the analogous operator $\mathcal{H}_1'$ in the CPT due to the absence of the term

$$(3\lambda/4\omega^2)(4\hat{n} + 2(a^+)^2 a^2 + 1)$$

introduced in the operator $\hat{\mathcal{H}}_0$. As a result the propagator $(\hat{\mathcal{H}}_0 - E_0^{(0)})^{-1}$, which defines the high-order corrections of the OM series, acts at an arbitrary intermediate n-quantum state as follows

$$(\hat{\mathcal{H}}_0 - E_0^{(0)})^{-1}|n\rangle = \frac{2\omega^2}{n}\frac{1}{\omega(\omega^2+1) + 3\lambda(n+1)}|n\rangle. \tag{23}$$

It differs from the CPT propagator by the coupling constant λ and a higher degree of n in the denominator.

The action of the operator $\mathscr{H}_1$ at the same state vector $|n\rangle$ is easily calculated

$$\hat{\mathscr{H}}_1|n\rangle = \frac{\lambda}{4\omega^2}\left(\frac{(n+4)!}{n!}\right)^{1/2}|n+4\rangle - \frac{1}{4\omega^2}[\omega(\omega^2-1)-\lambda(4n+6)]\left(\frac{(n+2)!}{n!}\right)^{1/2}|n+2\rangle$$
$$-\frac{1}{4\omega^2}[\omega(\omega^2-1)-\lambda(4n-2)]\left(\frac{n!}{(n-2)!}\right)^{1/2}|n-2\rangle$$
$$+\frac{\lambda}{4\omega^2}\left(\frac{n!}{(n-4)!}\right)^{1/2}|n-4\rangle \tag{24}$$

and contains the vertices corresponding to the creation or annihilation of two or four quanta. In order to simplify further consideration, let us introduce the graph notion for terms of the OM series which are calculated by means of the perturbation theory in respect of the operator (24). One can represent these terms using the elementary two- or four-quantum graphs, drawn in figure 2(a). For example, figures 2(b)-(d) show all graphs of the second, third and fourth order. Analytical expressions are correlated with graphs in conformity with the following rules.

(i) To each vertex corresponds the factor

$$-\frac{1}{4\omega^2}[\omega(\omega^2-1)-2\lambda(2n+3)][(n+1)(n+2)]^{1/2}$$

for a two-quantum vertex, and

$$\frac{\lambda}{4\omega^2}\left(\frac{(n+4)!}{n!}\right)^{1/2}$$

for a four-quantum one.

(ii) The number n for the definite vertex is equal to the quantity of lines crossed on the graph by the horizontal straight line drawn from this vertex.

(iii) The transition between two vertices corresponds with the propagator (23) with n, which is equal to the number of lines between these vertexes.

For example, the first graph in figure 2(b) corresponds to the following analytical formula

$$\frac{3}{16\omega^2}\frac{\lambda[\omega(\omega^2-1)-6\lambda][\omega(\omega^2-1)-14\lambda]}{[\omega(\omega^2+1)+9\lambda][\omega(\omega^2+1)+15\lambda]}.$$

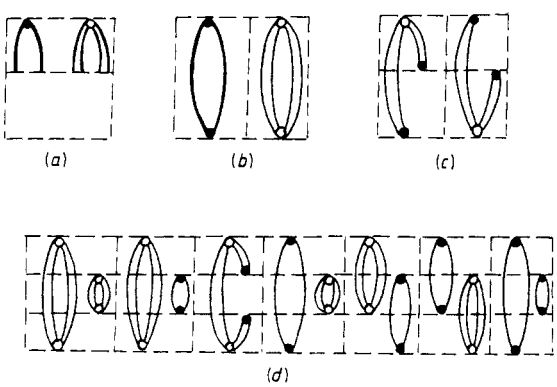

Figure 2. Graph presentation of the perturbation series for the QAO problem.

Let us now compare the behaviour of a sequence of graphs in the OM series and an analogous one in the CPT, when the number k of graph vertices is large ($k \gg 1$). It is well known (see, for example, the paper of Dolgov and Popov (1978)) that the CPT series is asymptotic and there are such sequences in it which diverge as $k!$ for $k \gg 1$. Figure 3(a) shows the graph of the $2k$-order term of one of such sequences.

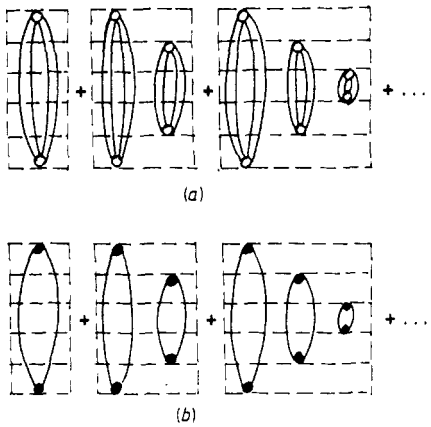

Figure 3. Sequences of graphs for the most divergent series in the CPT.

In the OM series this graph is described by the following expression

$$I_k = -\frac{\lambda^{2k}}{\omega^2}\frac{1}{2^{2k+1}}\frac{(4k)!4k[\omega(\omega^2+1)+3\lambda(4k+1)]}{(\prod_{m=1}^{k} 4m[\omega(\omega^2+1)+3\lambda(4m+1)])^2}.$$

Simple algebraic transformation leads to the estimation

$$|I_k| < \frac{\lambda^{2k}}{\omega^2}\frac{1}{2^{2k+1}}\frac{(4k)!4k3\lambda(4k+1)}{(\prod_{m=1}^{k} 4m3\lambda(4m+1))^2}$$

$$= \frac{24\lambda k(k+\frac{1}{4})}{\omega^2 2^{10k}3^{2k}}\frac{(4k)!\Gamma^2(\frac{5}{4})}{(k!)^2\Gamma^2(k+\frac{5}{4})} \approx \frac{48\lambda}{\omega^2}\frac{\Gamma^2(\frac{5}{4})}{(2\pi)^{3/2}}\frac{1}{(36)^k},$$

if $k \gg 1$.

The sequence of graphs considered converges absolutely as the geometric progression with denominator $q = (36)^{-1}$. At the same time this graph in the CPT doesn't contain the factor $[\Gamma(k+\frac{5}{4})]^2$ in the denominator and diverges at $\sim(k!)^2$.

We note that analogous investigation of the sequence of graphs from figure 3(b) shows that it converges as the geometric progression with denominator $q = \frac{4}{9}$. Unfortunately, we can't consider an infinite number of graphs for QAO but any real sequence of graphs proves to be absolutely converged and this apparently shows the absolute convergence of the whole of the OM series for an arbitrary parameter ω.

Now let us discuss the significance of choice of the optimal frequency ω. It is evident that if one defines ω by means of (5) it will lead to the disappearance of all graphs which begin or end with a two-quantum vertex. It means actually that we fulfil the partial summation of these graphs and take them into account in the zeroth-order approximation. Figure 2 shows that all graphs with odd numbers of vertices and about half of second- and fourth-order graphs will be equal to zero for the QAO problem

if the parameter ω is chosen on the basis of (5). It is essential that use of the frequency permits us to find the accurate asymptotical behaviour when $\lambda \gg 1$ (see our paper FK). The calculation of such dependence requires us to sum an infinite number of terms of the CPT series.

4. Integrals of motion

The OM permits us to take into account naturally the accurate laws of conservation in calculating the eigenvalues and eigenstates of the Hamiltonian. It is essential for such systems where integrals of motion qualitatively change the character of the energy spectrum. In many papers (see, for example, Bogoliubov (1950)) this problem was solved in such a way that eigenvalues of the system had been calculated approximately but the law of conservation had also been taken into account exactly in each approximation order. The introduction of the exact law of conservation to the approximate calculation scheme had essentially complicated all the calculations.

In contradiction to this approach the OM enables us to calculate the energy and to take into account the laws of conservation with the same accuracy and without any essential change of the calculation scheme. Feranchuk and Komarov (1982) considered the calculation of the Hamiltonian eigenvalues for the potential with two symmetrical minimums when the conservation of parity removed the degeneracy of the energy levels. In this paper we consider the more complicated case of the periodical potential when the energy spectrum has a zone structure. In particular, we use the OM for the potential

$$V(x) = h \cos 2x, \qquad V(x+\pi) = V(x),$$

because in this case the Schrödinger equation is reduced to the Mathieu equation and this permits us to compare our results with well known numerical solutions. The corresponding equation for the particle with 0.5 mass is

$$\hat{\mathscr{H}}\Psi_{nk} \equiv (-\mathrm{d}^2/\mathrm{d}x^2 + h\cos 2x)\Psi_{nk}(x) = E_n(k)\Psi_{nk}(x). \tag{25}$$

We note that if one changes $\Psi \to x$; $x \to t$ in equation (25) this equation will describe the problem of parametric classical oscillations. Therefore the method considered in the present work for investigation of the quantum system can also be used in the theory of oscillations of mechanical systems.

In accordance with the Bloch theorem, eigenfunctions of the Hamiltonian (25) are at the same time the eigenfunctions of the operator $\hat{T}$ of the wavefunction translation on the potential period

$$\hat{T}\Psi_{nk}(x) \equiv \Psi_{nk}(x+\pi) = \exp(\mathrm{i}k\pi)\Psi_{nk}(x). \tag{26}$$

As a result the particle energy spectrum has a zone structure $E \to E_n(k)$, where n is the zone number and k is the quasimomentum.

Now let us introduce the projection operator $\hat{\mathscr{P}}_k$ which transforms the arbitrary vector to the state with a definite quasimomentum value. One can easily verify that this operator is

$$\hat{\mathscr{P}}_k = \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}(k-\hat{p})m\pi]. \tag{27}$$

In order to find the common solution of (25) and (26) one can calculate the state

vector $|\psi\rangle$ which satisfies the following equation

$$\hat{\mathscr{L}}|\psi\rangle \equiv (\hat{\mathscr{H}} - E_n(k))\hat{\mathscr{P}}_k|\psi\rangle = 0, \tag{28}$$

and then the unknown state vector is

$$\psi_{nk} = \hat{\mathscr{P}}_k|\psi\rangle = \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}(k-\hat{p})m\pi]|\psi\rangle. \tag{29}$$

We shall solve equation (28) by means of the OM. Let us put the operator $\hat{\mathscr{L}}$ in second quantised form using the equation (1) with $u = 0$ and transform it to the normal form

$$\begin{aligned}\hat{\mathscr{L}} = \sum_{m=-\infty}^{\infty} & \exp[\mathrm{i}\pi mk - \tfrac{1}{4}\omega\pi^2 m^2 + (\omega/2)^{1/2}\pi m a^+] \\ & \times\{-\tfrac{1}{2}\omega[a^{+2} + [a + (\omega/2)^{1/2} m\pi]^2 - 2a^+(a + (\omega/2)^{1/2} m\pi) - 1] \\ & - E_n(k) - (-1)^m \tfrac{1}{2}h \exp(-1/\omega) \\ & \times[\exp(\mathrm{i}(2/\omega)^{1/2} a^+) \exp(\mathrm{i}(2/\omega)^{1/2} a) \\ & + \exp(-\mathrm{i}(2/\omega)^{1/2} a^+) \exp(-\mathrm{i}(2/\omega)^{1/2} a)]\} \exp((\omega/2)^{1/2}\pi m a).\end{aligned}$$

The operator $\hat{\mathscr{L}}_0$ of zeroth-order approximation is defined by that part of the operator $\hat{\mathscr{L}}$ which commutes with the particle number operator a^+a. The eigenstate of $\hat{\mathscr{L}}_0$ is equal to the vector $|n\rangle$ and the eigenvalue is defined by the following equation

$$\begin{aligned}\sum_{m=-\infty}^{\infty} & \exp(\mathrm{i}\pi mk - (\omega/4)\pi^2 m^2)\{(\omega/2)[(1-y)L_n(y) + 2nL_{n-1}(y) \\ & - 8yL^1_{n-1}(y) - 2yL^2_{n-2}(y)] - E^{(0)}_n(k)L_n(y) \\ & + h(-1)^m \exp(-1/\omega)L_n[y + (2/\omega)]\} = 0 \\ & y = \tfrac{1}{2}\omega\pi^2 m^2,\end{aligned} \tag{30}$$

where L^ν_n is the associated Laguerre polynomial; the number n defines the number of the zone and k is the quasimomentum eigenvalue.

Let us choose the parameter ω_{nk} from condition (5), that is put equal to zero the term $\sim a^{+2}$ in the operator $\hat{\mathscr{L}}$. This condition leads to the equation which we write for the ground-energy zone corresponding to the quantum number $n = 0$

$$\begin{aligned}\omega_{0k}\sum_m m^2\Omega_m = & \sum_m \left(\frac{1}{\pi^2} + \omega_{0k}\frac{\pi^2 m^2}{8}\right)\Omega_m \\ & + h\exp\left(-\frac{1}{\omega_{0k}}\right)\sum_m (-1)^m\left(\frac{1}{\pi^2\omega^2_{0k}} - \frac{m^2}{4}\right)\Omega_m \\ & - \frac{\omega^2_{0k}\pi^2}{8}\left(\sum_m m^2\Omega_m\right)^2\left(\sum_m \Omega_m\right)^{-1}; \\ & \Omega_m = \exp(-\tfrac{1}{4}\omega_{0k}\pi^2 m^2 + \mathrm{i}km).\end{aligned} \tag{31}$$

The common solution of (31) and (30) with $n = 0$ defines the zeroth approximation of the particle energy spectrum $E^{(0)}_0(k)$ in the ground zone. In considering the approximation, the wavefunction corresponding to these energy levels is

$$\psi_{0k} = \sum_{m=-\infty}^{\infty} \exp(-\tfrac{1}{4}\omega_{0k}\pi^2 m^2 + \mathrm{i}km + \mathrm{i}\pi m(\omega_{0k}/2)^{1/2} a^+)|0\rangle.$$

Thus, the calculation of the energy zone spectrum is reduced to the solution of two algebraic transcendental equations which can be easily programmed. We note that the sums in these equations converge very rapidly and one can calculate them using a desk calculator.

Table 3 compares the values $E_0^{(0)}(0)$ and $E_1^{(0)}(0)$ calculated by us on the basis of the OM zeroth approximation with the known numerical solutions of the Mathieu equation described, for example, by McLachlan (1949). Here $E_0^{(0)}(0)$ and $E_1^{(0)}(0)$ are the energy levels of zeroth and first zones with quasimomentum $k=0$. These levels correspond to the periodical solution of the Mathieu equation. This table shows that the OM zeroth approximation gives very high accuracy ($\sim 0.01\%$) and a sufficiently simple calculation of the algorithm. Also the regular character of the OM permits us to improve this accuracy by means of perturbation theory as regards the operator $\hat{\mathscr{L}} - \hat{\mathscr{L}}_0$. As an example we write out the formula for the value $E_0^{(2)}(0)$ (the first-order correction for energy of any level equals zero)

$$E_0^{(2)}(0) = -\frac{1}{\omega_0(\Sigma_m \Omega_m)} \sum_{k=3}^{\infty} \frac{1}{k(k-2)!}\left(\frac{Z_k^2}{k(k-1)} + \frac{k\omega_0^2 X_k^2}{(k-1)} + \omega_0^2 k(k-1) V_k^2\right),$$

$$Z_k = \sum_m \exp\left(-\frac{\omega}{4}\pi^2 m^2\right)\left\{\frac{\omega_0}{2}\left(1 - \frac{2E_0^{(0)}}{\omega_0} - \omega_0 \pi^2 m^2\right)\left[\pi m\left(\frac{\omega_0}{2}\right)^{1/2}\right]^k + \frac{h}{2}(-1)^m \exp\left(-\frac{1}{\omega_0}\right) \mathrm{Re}\left\{\left[\pi m\left(\frac{\omega_0}{2}\right)^{1/2} - \mathrm{i}\left(\frac{2}{\omega_0}\right)^{1/2}\right]^k\right\}\right.,$$

$$X_k = \sum_m \left[\pi m\left(\frac{\omega_0}{2}\right)^{1/2}\right]^k \exp\left(-\frac{\omega_0}{4}\pi^2 m^2\right),$$

$$Y_k = \sum_m \left[\pi m\left(\frac{\omega_0}{2}\right)^{1/2}\right]^{k-2} \exp\left(-\frac{\omega_0}{2}\pi^2 m^2\right).$$

In conclusion we note that the method considered may be very effective for the three-dimensional periodic potential due to essential decreasing of calculation time in comparison with direct solution of the Schrödinger equation.

Table 3. Eigenvalues of the Mathieu equation.

			h		
$-E$	10	20	30	40	50
$-E_0$ (acc.)	5.7916	13.937	22.513	31.313	40.257
$-E_0^{(0)}$	5.8002	13.933	22.510	31.311	40.254
$-E_1$ (acc.)	−1.8582	2.3991	8.1011	14.491	21.315
$-E_1^{(0)}$	−2.0153	2.3606	8.0806	14.475	21.301

5. New approach to the theory of nonlinear oscillations of Hamiltonian systems

At present the theory of nonlinear oscillations is an independent section of mathematical physics which has important applications in mechanics, radio techniques and astronomy. In spite of the great progress of computors the analytical theory of such systems remains

as before because it is necessary for qualitative analysis of the systems and for the development of the effective algorithms of numerical solutions.

The theory of nonlinear oscillations is as a rule reduced to the construction of the periodical solution of the following equation

$$\ddot{x}+x+\lambda f(x,\dot{x})=0, \qquad \dot{x}\equiv dx/dt. \tag{32}$$

Analytical methods known at present are based on expanding the solution $x(t,\lambda)$ of (32) in a power series of λ which proves to be asymptotic and doesn't permit one to analyse the systems in the region of sufficiently large λ and to develop the convergent algorithm of the numerical solution (see, for example, the work of Hayashi (1964)).

We suggest a new approach to the analytical theory of nonlinear oscillations of Hamiltonian systems based on the use of the OM for approximate solutions of the 'conjugate' Schrödinger equation. As will be shown for the real system of the type considered, our method permits one to find the solution of (32) with an accuracy of some percent in the whole range of $0\leqslant\lambda<\infty$ even in zeroth approximation. The higher approximations are constructed by the usual simple method and give a uniformly convergent series for $x(t,\lambda)$ for any $0\leqslant\lambda<\infty$. The results of this paragraph demonstrate the fact that the OM gives a good approximation not only for eigenvalues, but also for eigenfunctions.

The scheme of calculations in the method considered is as follows. Let us suppose that the physical system described by (32) is Hamiltonian; i.e. this equation can be obtained from the Lagrange variational principle. Then one can as usual construct the Hamiltonian of such nonlinear classical system and introduce the 'conjugate' quantum system by means of the formal parameter $\hbar$ and substitute instead of x and p the corresponding operators. Eigenfunctions and eigenvectors of the Hamiltonian obtained are calculated by means of the OM being applied for any λ. In the last stage of the calculation we fulfil the transition to the classical limit $\hbar\to 0$ and find the law of motion $x(t,\lambda)$. It is essential that this transition is reduced to the solution of the algebraic equation the general structure of which doesn't depend on the real potential form. Thus, we use the quantum mechanical approach for analysis of classical system just as the inverse consideration is usually used.

Let us introduce the main features of our method on the example of anharmonic oscillator which permits us to compare our results with an accurate analytical theory. This mechanical system is described by the following equation

$$\ddot{x}+x+4\lambda x^3=0. \tag{33}$$

This equation of motion corresponds to the 'conjugate' quantum system with Hamiltonian

$$\hat{\mathscr{H}}=\tfrac{1}{2}(\hat{p}^2+\hat{x}^2)+\lambda\hat{x}^4, \qquad \hat{p}=-i\hbar(d/dx),$$

whose eigenvalues and eigenstates were found in § 2:

$$\varepsilon_n^{(0)}=\hbar\{\tfrac{1}{4}[\omega_n+(1/\omega_n)](2n+1)+(3\lambda\hbar/4\omega_n^2)(1+2n+2n^2)\}, \tag{34}$$

where we left parameter $\hbar$ in the equation and ω_n is defined by the following equation

$$\omega_n^3-\omega_n-2\lambda\hbar(2n+3)=0. \tag{35}$$

Now let us find the initial equation (33) solution which is the quasiclassical limit of the 'conjugate' quantum problem. In particular, the period of the classical motion

is defined by the well known expression

$$\Omega = 2\pi/T = (1/\hbar)(\partial\varepsilon_n/\partial n);$$
$$\hbar \to 0, \qquad n \to \infty, \qquad \hbar n = \beta = \text{const}, \tag{36}$$

which leads, in the approximation considered, to the following result

$$\Omega = \frac{\omega}{2}\frac{7\omega^2 - 3}{3\omega^2 - 1}$$

and $\omega(\beta)$ is the solution of the equation

$$\omega^3 - \omega - 4\lambda\beta = 0, \tag{37}$$

where parameter β is connected with the total energy E

$$E = \tfrac{1}{8}\beta[7\omega(\beta) + (1/\omega(\beta))]. \tag{38}$$

The system of algebraic equations (36)-(38) defines the zeroth approximation for period of the classical anharmonic oscillator in dependence on the total energy

$$\omega(E) = [[\tfrac{3}{7}\{1 + [1 + \tfrac{7}{9}(32\lambda E + 1)]^{1/2}\}]]^{1/2},$$
$$T(E) = \frac{4\pi}{\omega}\frac{3\omega^2 - 1}{7\omega^2 - 3}. \tag{39}$$

It is well known, that the accurate expression of the function $T(E)$ is defined by the full elliptic integral

$$T(E) = \frac{4}{\sqrt{u}}K\left(\frac{u-1}{2u}\right), \qquad u = (1 + 16\lambda E)^{1/2}$$

and table 4 shows that simple formula (39) gives a good approximation of the function $T(E)$ in the whole range of changing parameter λ and total oscillator energy E.

Table 4. The particle oscillation period and the law of motion in the potential $\frac{1}{2}x^2 + \lambda x^4$.

			λE		$(\lambda E)^{1/4}T$,
T	0.1	0.5	1.0	10	$\lambda E \gg 1$
$T(E)$ (acc.)	5.211 98	4.004 31	3.473 06	2.046 04	3.708 15
$T_2(E)$	5.211 95	4.004 25	3.473 02	2.046 05	3.708 20
			$\tau = 2(\lambda E)^{1/4}t$, $\lambda E \gg 1$		
x	0	0.4	1.0	1.4	1.6
$x(\tau)$ (acc.)	1	0.923	0.596	0.321	0.176
$x(\tau)$ (44)	1.017	0.905	0.551	0.293	0.164
$x_1(\tau)$	1.001	0.928	0.577	0.312	0.171

One can calculate the function $T(E)$ more precisely taking into account the second-order corrections to energy levels ε_n. The calculation will be essentially simplified if we choose the parameter ω_n in this approximation from the condition that the second-order correction $\varepsilon_n^{(2)}$ would be equal to zero in the limit $n \to \infty$, $\hbar \to 0$. This

condition leads to the equation for $\omega(\beta)$:

$$(\lambda\beta)^2\left(\frac{3\lambda\beta}{\omega(\omega^2+1)+6\lambda\beta}-2\right)+[\omega(\omega^2-1)-4\lambda\beta]$$
$$\times\left(8\lambda\beta-\omega(\omega^2-1)+\frac{3\lambda\beta[\omega(\omega^2-1)-4\lambda\beta]}{\omega(\omega^2+1)+6\lambda\beta}\right)=0 \tag{40}$$

and the oscillator energy is defined by the same equation (34). Table 4 shows the period $T_2(E)$ calculated by means of (40) and it proves that T_2 and T coincide with very high accuracy.

Now consider the calculation of classical law of motion. The solution $x(t)$ of (32) we find as the quasiclassical limit ($\hbar\to 0$, $n\to\infty$, $\hbar n=\beta$) of the average value of an operator $\hat{x}$ on some state vector $|\psi(t)\rangle$, that is

$$x(t)=\sum_{m,n} c_m^* c_n\langle\psi_m|\hat{x}|\psi_n\rangle\exp\left[\frac{1}{\hbar}(E_m-E_n)(t-t_0)\right] \tag{41}$$

where $|\psi_n\rangle$ and E_n are the accurate eigenfunctions and eigenvalues of Hamiltonian of the 'conjugate' quantum system and c_n are the coefficients defining the initial wave packet.

The classical trajectory of particle corresponds to such a wave packet in which the coefficients c_n have a sharp maximum near the value $n_0\gg 1$ defined by the conditions

$$E_{n_0}=E, \qquad n_0\hbar=\beta.$$

Therefore the law of motion which does not depend on the wave packet form is

$$x(t)=\sum_{k=-\infty}^{\infty}\langle\psi_{n_0+k}|\hat{x}|\psi_{n_0}\rangle\exp[\Omega k(t-t_0)],$$
$$\Omega=\frac{1}{\hbar}\frac{\partial E_n}{\partial n}\bigg|_{n_0}. \tag{42}$$

This function contains two arbitrary constants E and t_0 but later we shall put $t_0=0$.

Now use the OM approximate expressions for the state vectors $|\psi_n\rangle\approx|n,\omega_n\rangle$ and energy levels $E_n\simeq\varepsilon_n^{(0)}$. It is necessary to take into account that the function $|n,\omega_n\rangle$ and $|m,\omega_m\rangle$ are non-orthogonal because they correspond to different frequencies. For this reason (42) is transformed as follows

$$x(t)=\frac{\sum_{k=-\infty}^{\infty}\langle n_0+k,\omega_{n_0+k}|\hat{x}|n_0,\omega_{n_0}\rangle\exp(ik\Omega t)}{\sum_{k=-\infty}^{\infty}\langle n_0+k,\omega_{n_0+k}|n_0,\omega_{n_0}\rangle\exp(ik\Omega t)}. \tag{43}$$

Both sums in (43) can be calculated analytically in the limit $n_0\to\infty$. This calculation does not depend on the real form of the energy spectrum ε_n and it is fulfilled most compactly when using the following operator equality (see (19))

$$\exp[\xi(\omega,\omega')\hat{A}(\omega')]a(\omega')\exp[-\xi(\omega,\omega')\hat{A}(\omega')]=a(\omega);$$
$$\xi(\omega,\omega')=\tfrac{1}{4}\ln(\omega'/\omega), \qquad A(\omega')=[a^+(\omega')]^2-[a(\omega')]^2,$$

where $a(\omega)$ is the annihilation operator attributed to the state vector with frequency ω. This equality permits us to calculate an arbitrary matrix element in (43) and to find

the following expression

$$\langle n+k, \omega_{n+k} | n+s, \omega_n \rangle_{n\gg 1} = \begin{cases} \mathcal{J}_{(k-s)/2}(\gamma k) & \text{if } (k-s) \text{ is an even integer,} \\ 0 & \text{if } (k-s) \text{ is an odd integer.} \end{cases}$$

Here $\mathcal{J}_\nu(x)$ is the Bessel function;

$$\gamma = (\beta/4\omega(\beta))(\partial\omega(\beta)/\partial\beta)$$

and the parameter β was defined above.

In order to calculate the sums over all k one has to use the integral presentation of the Bessel function

$$\mathcal{J}_\nu(\gamma k) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\varphi \cos(\gamma k \sin\varphi - k\varphi).$$

As a result we find the law of motion $x(t)$ in the following form:

$$x(t) = (2\beta/\Omega(\beta))^{1/2} \cos[\tfrac{1}{2}\varphi(t)], \tag{44}$$

where $\varphi(t)$ is defined by the algebraic equation

$$\Omega t + \gamma \sin\varphi(t) - \tfrac{1}{2}\varphi(t) = 0. \tag{45}$$

We see, that equations (44) and (45) have a universal form which does not depend on the real potential type. It is natural that these equations give the same results as any asymptotic method if the nonlinear parameter $\lambda \ll 1$. But table 4 shows that our method gives the solution which coincides sufficiently closely with the accurate classical solution even in the most unfavourable case $\lambda E \gg 1$. The coincidence is essentially improved when the OM corrections to the energies and wavefunctions of steady states are taken into account

$$E_n = \varepsilon_n^{(0)} + \varepsilon_n^{(2)}, \qquad |\psi_n\rangle \approx |n, \omega_n\rangle + |\psi_n^{(1)}\rangle.$$

For example, the calculation of such corrections for the anharmonic oscillator in the $\lambda E \gg 1$ case leads to the following law of motion

$$x_1(t) = (2\beta/\Omega(\beta))^{1/2} \tfrac{1}{192}(188\cos\Omega t + 13\cos 3\Omega t - \cos 5\Omega t),$$

which is also presented in table 4.

Thus the results obtained in this paragraph show that our approach permits us to reduce the solution of a *nonlinear* classical problem to the solution of a *linear* quantum problem.

6. Description of many-dimensional systems

In the preceding sections we have demonstrated the efficiency of the OM for one-dimensional systems. However, one could already understand that the general scheme of the OM was suitable for systems with any number of degrees of freedom. In order to prove the latter affirmation, we investigate two many-dimensional problems.

Let us first consider the two-dimensional problem of coupled quartic anharmonic oscillators investigated numerically by Hioe *et al* (1978). The most simple form of the Hamiltonian of the system is

$$\hat{\mathcal{H}} = \tfrac{1}{2}(p_1^2 + y_1^2) + \tfrac{1}{2}(p_2^2 + y_2^2) + \lambda(y_1^4 + 2by_1^2y_2^2 + y_2^4), \tag{46}$$

where $y_{1,2}$ and $p_{1,2}$ are the coordinate and momentum operators of the particles; λ and b are the dimensionless parameters of the Hamiltonian. The operator (46) can be put in second quantised form through the introduction of creation and annihilation operators

$$y_{1,2}=[1/(2\omega_{1,2})^{1/2}](a_{1,2}+a_{1,2}^{+}), \qquad p_{1,2}=\mathrm{i}(\omega_{1,2}/2)^{1/2}(a_{1,2}^{+}-a_{1,2}) \tag{47}$$

with arbitrary parameters $\omega_{1,2}$.

Let us put this Hamiltonian in the normal form. Then it can be divided into two parts in conformity with the OM scheme

$$\hat{\mathscr{H}}=\hat{\mathscr{H}}_0+\hat{\mathscr{H}}_1$$

$$\hat{\mathscr{H}}_0=\sum_{\nu=1,2}\left[\frac{1}{2}\left(\omega_\nu+\frac{1}{\omega_\nu}\right)\left(a_\nu^{+}a_\nu+\frac{1}{2}\right)+\frac{3\lambda}{4\omega_\nu^2}(1+2a_\nu^{+}a_\nu+2(a_\nu^{+}a_\nu)^2)\right]+\frac{\lambda b}{2\omega_1\omega_2}(1+2a_1^{+}a_1)(1+2a_2^{+}a_2), \tag{48}$$

$$\begin{aligned}\hat{\mathscr{H}}_1=\sum_{\nu=1,2}\Bigg[&-\frac{1}{4}\left(\omega_\nu-\frac{1}{\omega_\nu}\right)(a_\nu^{+2}-a_\nu^2)\\&+\frac{\lambda}{4\omega_\nu^2}(6a_\nu^{+2}+6a_\nu^2+a_\nu^{+4}+a_\nu^4+4a_\nu^{+3}a_\nu+4a_\nu^{+}a_\nu^3)\Bigg]\\&+\frac{\lambda b}{2\omega_1\omega_2}[(1+2a_1^{+}a_1)(a_2^{+2}+a_2^2)+(1+2a_2^{+}a_2)(a_1^{+2}+a_1^2)\\&+a_1^{+2}a_2^{+2}+a_1^{+2}a_2^2+a_2^{+2}a_1^2+a_1^2a_2^2].\end{aligned} \tag{49}$$

The Hamiltonian $\hat{\mathscr{H}}_0$ commutes with the particle number operators $\hat{n}_{1,2}=a_{1,2}^{+}a_{1,2}$ and therefore the zeroth-order approximation for eigenstates is given by the equations

$$|\psi_{n_1n_2}^{\pm}\rangle=(1/\sqrt{2})(|n_1n_2\rangle\pm|n_2n_1\rangle), \qquad \hat{n}_{1,2}|n_1n_2\rangle=n_{1,2}|n_1n_2\rangle, \tag{50a}$$

when n_1 and n_2 are both odd or both even numbers and

$$|\psi_{n_1n_2}\rangle=|n_1n_2\rangle \tag{50b}$$

in the opposite case.

Then one can find the following approximate equation for the Hamiltonian eigenvalues

$$E_{n_1n_2}^{(0)}=\sum_{\nu=1,2}\left[\frac{1}{2}\left(\omega_\nu+\frac{1}{\omega_\nu}\right)\left(n_\nu+\frac{1}{2}\right)+\frac{3\lambda}{4\omega_\nu^2}(1+2n_\nu+2n_\nu^2)\right]+\frac{\lambda b}{2\omega_1\omega_2}(2n_1+1)(2n_2+1). \tag{51}$$

The parameters $\omega_{1,2}^{(0)}$ are defined in such a way that the parts proportional to a_1^{+2} and a_2^{+2} in the operator $\hat{\mathscr{H}}_1$ would be equal to zero. These conditions lead to the following equations

$$\begin{aligned}&\left(\frac{1}{\omega_1}-\omega_1\right)+\frac{2\lambda}{\omega_1^2}(3+2n_1)+\frac{2\lambda b}{\omega_1\omega_2}(2n_1+1)=0,\\&\left(\frac{1}{\omega_2}-\omega_2\right)+\frac{2\lambda}{\omega_2^2}(3+2n_2)+\frac{2\lambda b}{\omega_1\omega_2}(2n_2+1)=0.\end{aligned} \tag{52}$$

Equations (51) and (52) permit us to calculate $E^{(0)}_{n_1 n_2}$ for any energy level and parameters λ and b. Table 5 shows some of these calculation results. It is clear that the zeroth-order approximation doesn't give very high precision, but the simplicity of the formulae and their applicability in the whole range of the Hamiltonian parameters is of great interest.

Table 5. Energy levels of the coupled quartic anharmonic oscillators.

	λ				
	0.1		1.0		10
	b				
E	1	−1	1	−1	−1
E_{00} (acc.)	1.1502	1.0813	1.7242	1.4438	2.5577
$E^{(0)}_{00}$	1.1527	1.0855	1.5658	1.4921	2.7707
$E^{(0)}_{00}+E^{(2)}_{00}$	1.1502	1.0813	1.7251	1.4444	2.5592
E_{10} (acc.)	2.4143	2.2120	3.8304	3.0666	5.4881
$E^{(0)}_{10}$	2.4196	2.2317	3.8357	3.1053	5.9247

In order to calculate eigenvalues $E_{n_1 n_2}$ more precisely, one can use the perturbation theory with respect to $\hat{\mathscr{H}}_1$. Corresponding formulae in the case of the ground state are

$$|\psi^{(1)}_0\rangle = -(\hat{\mathscr{H}}_0 - E_0)^{-1}\hat{\mathscr{H}}_1|0\rangle, \qquad a_1|0\rangle = a_2|0\rangle = 0,$$
$$E^{(2)}_0 = -\langle 0|\hat{\mathscr{H}}_1(\hat{\mathscr{H}}_0 - E_0)^{-1}\hat{\mathscr{H}}_1|0\rangle.$$

The perturbation theory precision is essentially increased if the value $E^{(0)}_0 + E^{(2)}_0$ is calculated without the zeroth-order parameters $\omega^{(0)}_{1,2}$, but with arbitrary $\omega_{1,2}$. Then the equations for $\omega_{1,2}$ are found from the condition that parts proportional to $a^{+2}_{1,2}$ in the operator $\hat{\mathscr{H}}^{(1)}_1$ should be equal to zero. The operator $\hat{\mathscr{H}}^{(1)}_1$ is defined by the formula (6)

$$\hat{\mathscr{H}}^{(1)}_1|0\rangle \equiv \hat{\mathscr{H}}_1|\psi^{(1)}_0\rangle = \hat{\mathscr{H}}_1[1-(\hat{\mathscr{H}}_0 - E_0)^{-1}\hat{\mathscr{H}}_1]|0\rangle.$$

This scheme leads to the following system of algebraic equations for the ground state energy ($n_1 = n_2 = 0$, $\omega_1 = \omega_2 = \omega$)

$$E^{(0)}_0 + E^{(2)}_0 = \frac{1}{2\omega^2}[\omega^3 + \omega + \lambda(3+b)] - \frac{1}{4\omega^2}\frac{[\omega(\omega^2-1)-2\lambda(3+b)]^2}{\omega(\omega^2+1)+3\lambda(3+b)}$$
$$-\frac{3\lambda}{2\omega^2}\frac{\omega(\omega^2+1)+\lambda(9+6b-b^2)}{B-\lambda^2 A} - \frac{\lambda^2 b^2}{2\omega^2}\frac{\omega(\omega^2+1)+2\lambda(6+b)}{B-\lambda^2 A},$$

$$B = [\omega(\omega^2+1)+4\lambda(3+b)]^2, \qquad A = \frac{5}{2}\left(\omega + \frac{1}{\omega}\right) + \frac{63\lambda}{2\omega^2} + \frac{9\lambda b}{2\omega^2}, \tag{53}$$

$$\omega(\omega^2-1) - 2\lambda(3+b) - \tfrac{3}{2}\lambda[\omega(\omega^2-1)-2\lambda(7+b)][\omega(\omega^2+1)+\lambda(9+6b-b^2)]$$
$$-\tfrac{1}{2}\lambda b[\omega(\omega^2-1)-2\lambda(3+5b)][\omega(\omega^2+1)+2\lambda(6+b)]/(B-\lambda^2 A).$$

Table 5 lists the results obtained by means of equations (53) and shows that the OM gives a uniformly convergent series for any λ and b.

Let us now consider another many-dimensional problem where simple formulae, obtained by means of the OM, are of practical interest, namely the problem of the hydrogen atom in a uniform magnetic field of arbitrary magnitude. The Hamiltonian

of the system is well known (see, for example, the paper of Simola and Virtamo (1978)).

$$\hat{\mathcal{H}}|\psi\rangle \equiv \left(\frac{1}{2}\hat{p}^2 - \frac{1}{r} + \gamma l_z + \frac{1}{2}\gamma(x^2+y^2)\right)|\psi\rangle = E|\psi\rangle, \tag{54}$$

where l_z is the z-projection of the angular momentum, the magnetic field $\boldsymbol{B}$ is directed along the z-axis and its magnitude is measured in units of B_0, that is

$$B = \gamma B_0, \qquad B_0 = \tfrac{1}{2}me^3c\hbar^{-3}.$$

Introduce annihilation and creation operators in accordance with the rule

$$x_\nu = [1/(2\omega_1)^{1/2}](a_\nu + a_\nu^+), \qquad \nu = 1, 2, \qquad x_3 = [1/(2\omega_2)^{1/2}](a_3 + a_3^+), \qquad \omega_1 \neq \omega_2.$$

In order to put Hamiltonian (54) in the second quantised form one can use the following integral presentation

$$\tau^{-1} = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\alpha^2\tau^2)\,\mathrm{d}\alpha$$

and take into account that $\hat{I}_2$ commutes with $\hat{\mathcal{H}}$. The eigenfunctions of the particle number operator are used in the zeroth-order approximation of the OM and therefore the operators a_ν and a_ν^+ have to transform in such a manner that $\hat{I}_z$ would be diagonal

$$a_1 = (b_1 + \mathrm{i}b_2)/\sqrt{2}, \qquad a_2 = (\mathrm{i}b_1 + b_2)/\sqrt{2}, \qquad a_3 = b_3.$$
$$\hat{I}_z = \hat{N}_- = b_1^+ b_1 - b_2^+ b_2, \qquad \hat{N}_+ = b_1^+ b_1 + b_2^+ b_2, \qquad N_3 = b_3^+ b_3. \tag{55}$$

Then the Hamiltonian (54) in the normal form is

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2}\left(\omega_1 + \frac{\gamma^2}{\omega_1}\right)(N_+ + 1) + \frac{\mathrm{i}}{2}\left(\omega_1 - \frac{\gamma^2}{\omega_1}\right)(b_1^+ b_2^+ - b_1 b_2) + \gamma N_- \\ & + \frac{\omega_2}{4}(2N_3 + 1 - b_3^{+2} - b_3^2) - 2\left(\frac{\omega_1}{\pi}\right)^{1/2}\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)(\beta x^2+1)^{1/2}} \\ & \times \exp[\varphi_1(x) b_1^+ b_2^+ + \varphi_2(x) b_3^+ b_3]\exp[f_1(x)N_+ + f_2(x)N_3] \\ & \times \exp[\varphi_2(x) b_3 b_3 - \varphi_1(x) b_1 b_2]\end{aligned} \tag{56}$$

where

$$\varphi_1 = \frac{\mathrm{i}x^2}{x^2+1}, \qquad \varphi_2 = \frac{\beta x^2}{2(\beta x^2+1)}, \qquad f_1 = -\ln(x^2+1)$$
$$f_2 = -\ln(\beta x^2+1), \qquad \beta = \omega_1/\omega_2.$$

Let us include in accordance with the OM scheme all terms, which commute with the operators $\hat{N}_\pm$, $\hat{N}_3$, in the zeroth-order Hamiltonian $\hat{\mathcal{H}}_0$. Then eigenfunctions and eigenvalues are defined by the quantum numbers $N_\pm$ and N_3 and energies of the ground state $E_{0+}^{(0)}(N_\pm = N_3 = 0)$ and the first odd state $E_{0-}^{(0)}(N_\pm = 0, N_3 = 1)$ are

$$E_{0+}^{(0)} = \tfrac{1}{2}(\omega_1 + \gamma^2/\omega_1) + (\omega_1/4\beta) - 2(\omega_1/\pi(\beta-1))^{1/2}\ln[\sqrt{\beta} + (\beta-1)^{1/2}], \tag{57}$$

$$\begin{aligned}E_{0-}^{(0)} = {} & \tfrac{1}{2}(\omega_1 + \gamma^2/\omega_1) + (3\omega_1/4\beta) \\ & - 2(\omega_1/\pi)^{1/2}(\beta-1)^{-3/2}\{\beta(\beta-1) - \ln[\sqrt{\beta} + (\beta-1)^{1/2}]\}.\end{aligned} \tag{58}$$

The parameters $\omega_{1\pm}$ and $\beta_\pm$ may be defined in this case both by means of (5) and from the conditions

$$\partial E^{(0)}_{0\pm}/\partial\omega_{1\pm} = \partial E^{(0)}_{0\pm}/\partial\beta_\pm = 0,$$

which lead to the same equations

$$\begin{aligned}
&\omega_{1+} = \frac{4}{\pi z}\left(\frac{z+1}{z-1}\right)^6 [\tfrac{1}{2}(z+1)\ln z - z + 1]^2,\\
&\gamma^2 = \omega_{1+}^2\left[\frac{1+4z+z^2}{(z+1)^2} - 2\left(\frac{z}{\pi\omega_{1+}}\right)^{1/2}\frac{\ln z}{z-1}\right],
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\omega_{1-} = \frac{4}{9\pi z}\frac{(z+1)^6}{(z-1)^{10}}[3(z+1)(z^2-1) - 2(z-1)^3 - 6z(z+1)\ln z]^2,\\
&\gamma^2 = \omega_{1-}^2\left[\frac{1+8z+z^2}{(z+1)^2} - 4\left(\frac{z}{\pi\omega_{1-}}\right)^{1/2}\frac{z^2-1-2z\ln z}{(z-1)^3}\right],\\
&z = [\sqrt{\beta} + (\beta-1)^{1/2}]^2.
\end{aligned} \tag{60}$$

We note that it is more convenient to calculate the functions $E^{(0)}(\gamma)$ from equations (59) and (60) in the parametric form as $E^{(0)}(z)$ and $\gamma(z)$. Table 6 shows a good conformity of the results, obtained on the basis of these simple formulae, with known results found by more complicated methods.

Table 6. Energy levels of the hydrogen atom in the uniform magnetic field.

	$\gamma = B/B_0$				
$\varepsilon = E - \gamma$	0.15	1.0	2.5	50	100
$\lvert\omega_{0+}\rvert$ (acc.)	0.628	1.02	1.38	3.76	4.72
$\lvert\varepsilon^{(0)}_{0+}\rvert$ (57)	0.562	0.956	1.31	3.60	4.49
$\lvert\varepsilon^{(0)}_{0+} + \varepsilon^{(2)}_{0+}\rvert$	0.598	0.995	1.35	3.75	4.72
$\lvert\varepsilon^{(0)}_{0+}\rvert$ (64)	0.629	0.993	0.988	—	—
$\lvert\varepsilon^{(0)}_{0+} + \varepsilon^{(2)}_{0+}\rvert_m$	0.628	1.02	1.32	—	—
$\lvert\varepsilon_{0-}\rvert$ (acc.)	—	0.298	—	—	0.476
$\lvert\varepsilon^{(0)}_{0-}\rvert$ (58)	0.192	0.279	0.323	0.408	0.455

In order to improve the energy estimation let us calculate the second-order approximation $E^{(2)}_{0+}$. Unlike the anharmonic oscillator the perturbation operator is not polynomial in the considered case and $E^{(2)}_{0+}$ contains infinite summation over intermediate states. This sum proves to be rapidly convergent and the main contribution is defined by four quantum states. In this approximation the correction $E^{(2)}_{0+}$ is

$$\begin{aligned}
E^{(2)}_{0+} = \frac{\omega_1}{\pi}\Bigg(&\frac{2[\mathcal{T}^{(2)}_{20}]^2}{(\omega_1/\pi)^{1/2}(4\mathcal{T}^{(2)}_{40} + \mathcal{T}^{(4)}_{40} + \mathcal{T}^{(0)}_{40} - \mathcal{T}^{(0)}_{00}) - \omega_1 - \gamma^2/\omega_1}\\
&+\frac{\frac{3}{4}\beta^4[\mathcal{T}^{(2)}_{02}]^2}{(\omega_1/\pi)^{1/2}(3\beta^2\mathcal{T}^{(2)}_{04} + 3\beta^4\mathcal{T}^{(4)}_{04} + \mathcal{T}^{(0)}_{04} - \mathcal{T}^{(0)}_{00}) - \omega_1/\beta}\\
&+\frac{2\beta^2[\mathcal{T}^{(2)}_{11}]^2}{(\omega_1/\pi)^{1/2}(\beta^2\mathcal{T}^{(4)}_{22} + \mathcal{T}^{(0)}_{22} - \mathcal{T}^{(0)}_{00} + \mathcal{T}^{(2)}_{22} + \frac{1}{2}\beta^2\mathcal{T}^{(2)}_{22}) - \omega_1 - \gamma^2/\omega_1 - \omega_1/\beta}\Bigg),
\end{aligned}$$

$$\mathcal{T}_{mn}^{(k)} = \int_0^\infty \frac{x^{2k}(\beta x^2+1)^{1/2}}{(1+x^2)^{m+1}(1+\beta x^2)^{n+1}}\,dx,$$

and its numerical values are also listed in table 6.

Let us now consider the same problem of the hydrogen atom in the magnetic field but using the Hamiltonian (54) when the algebraic structure is essentially simplified. Let us use the coordinate transformation introduced by Komarov and Romanova (1982) and considered by us already for the Hamiltonian (11). In these variables the Schrödinger equation for the system considered is transformed as follows:

$$\begin{aligned}(\hat{\mathcal{H}}_1-\varepsilon)\psi(\xi_1\xi_2) \equiv \Bigg\{ &-\frac{1}{2}\left(\frac{\partial^2}{\partial\xi_1^*\partial\xi_1}+\frac{\partial^2}{\partial\xi_2^*\partial\xi_2}\right)\\ &+(\xi_1^*\xi_1+\xi_2^*\xi_2)\left[\frac{1}{2}\gamma\left(\xi_1^*\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}-\xi_1\frac{\partial}{\partial\xi_1}-\xi_2^*\frac{\partial}{\partial\xi_2^*}\right)\right.\\ &\left.+2\gamma^2|\xi_1|^2|\xi_2|^2-E\right]-\varepsilon\Bigg\}\psi(\xi_1\xi_2)=0\end{aligned} \tag{61}$$

and the nucleus charge $z=\varepsilon$ is a part of the eigenvalue. One has to find those solutions of (61) which corresponds to $\varepsilon=1$. According to Komarov and Romanova (1982) this equation is equivalent to the Schrödinger equation in the real three-dimensional space when the function $\psi(\xi_1\xi_2)$ conforms to zeroth eigenvalue of the following operator

$$\hat{Q}=\xi_1^*\frac{\partial}{\partial\xi_1^*}+\xi_2^*\frac{\partial}{\partial\xi_2^*}-\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2} \tag{62}$$

commuted with Hamiltonian (54) and operator $\hat{l}_z$. As a consequence, the function $\psi(\xi_1\xi_2)$ doesn't depend on the angle

$$\varphi=\tan^{-1}[\mathrm{i}(\xi_1^*-\xi_1)/(\xi_1^*+\xi_1)].$$

Now we use the operators a_s^+, a_s and b_s, b_s^+ introduced earlier in Hamiltonian (11). Then the zeroth-order Hamiltonian $\hat{\mathcal{H}}_0$ commutes with the operator $\hat{Q}$, the particle number operator and is

$$\begin{aligned}\hat{\mathcal{H}}_0=\sum_{s=1,2}\Bigg[&\left(\frac{\omega_s}{4}-\frac{E}{2\omega_s}\right)(1+\hat{N}_s+\hat{M}_s)-\frac{\gamma}{\omega_s}(-1)^s(\hat{N}_s-\hat{M}_s+\hat{N}_s^2-\hat{M}_s^2)\Bigg]\\ &+\frac{\gamma}{4(\omega_1\omega_2)^{1/2}}[(\hat{N}_1-\hat{M}_1)(1+\hat{N}_2-\hat{M}_2)-(\hat{N}_1+1-\hat{M}_2)(\hat{N}_2-\hat{M}_2)]\\ &+\frac{\gamma^2}{4\omega_1^2\omega_2}(2+3\hat{N}_1+3\hat{M}_1+\hat{N}_1^2+\hat{M}_1^2+4\hat{N}_1\hat{M}_1)(1+\hat{N}_2+\hat{M}_2)\\ &+\frac{\gamma^2}{4\omega_1\omega_2^2}(2+3\hat{N}_2+3\hat{M}_2+\hat{N}_2^2+\hat{M}_2^2+4\hat{N}_2\hat{M}_2)(1+\hat{N}_1+\hat{M}_1).\\ &\hat{N}_s=a_s^+a_s, \qquad \hat{M}_s=b_s^+b_s.\end{aligned} \tag{63}$$

The zeroth-order energy $E_{0+}^{(0)}$ of the ground state ($a_s|0\rangle=b_s|0\rangle=0$) is calculated from the condition $\varepsilon^{(0)}=1$ and the parameter ω is defined by means of the above-

mentioned scheme. This leads to the following equations

$$\omega/2 - E_{0+}^{(0)}/\omega + \gamma^2/\omega^3 = \varepsilon = 1,$$
$$\omega/4 + E_{0+}^{(0)}/2\omega - 3\gamma^2/2\omega^3 = 0. \tag{64}$$

Table 6 shows that formulae (64) give very high precision because the Coulomb singularity is taken into account in Hamiltonian (61) exactly. The considered Hamiltonian is a polynomial with respect to a_s, b_s and calculation of the second-order correction is a simple algebraic procedure. These results are also shown in the table 6.

In conclusion we note that recently Feranchuk *et al* (1984) considered the polaron problem on the basis of the OM and showed that this method permitted one to investigate in detail the systems with infinite number of degrees of freedom.

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